THE UNIVERSITY OF ALBERTA

HOMOTOPY AND FIBRATIONS IN CATEGORIES

BY

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ABSTRACT

The purpose of this thesis is to study notions of homotopy and fibrations in arbitrary categories. The basic approach taken is that of homotopy systems first defined by Kan. For topological like categories homotopy systems have been studied by Kamps via semi-cubical complexes. Other standard approaches taken to homotopy include those of Bauer and Dugundji via classes of morphisms, which we compare with homotopy systems and those of Ringel via diagonalizable pairs of morphisms.

Most of chapter I consists of a brief introduction to category theory including definitions, terminology and some basic categorical results needed in the remainder of this thesis. The chapter ends with an introduction to homotopy and fibrations in a general categorical sense via natural equivalence relations. Some results on these ideas are given.

Chapter II consists of an introduction to homotopy systems and the resulting homotopy relations, fibrations and cofibrations following the method of Kamps. Basic properties are given and many examples, some new, are explained to show the generalities and potential of taking homotopy systems as the "proper" approach to the study of abstract homotopy.

Chapter III is devoted entirely to a systematic development of the Eckmann-Hilton injective homotopy theory for modules in terms of a homotopy system. We show that we do regain the usual injective homotopy relation and the injective

fibrations from this homotopy system.

The categories of fractions, M - homotopy and M - fibrations of Bauer and Dugundji are introduced in chapter IV. Relationships between categories of fractions and M - homotopy categories are developed and also between M - homotopy categories and the homotopy categories determined from homotopy systems. Similarly M - fibrations and fibrations from homotopy systems are compared.

In chapter V cohomotopy systems are defined and relationships between homotopy systems and cohomotopy systems are developed. In particular when the two functors involved are adjoint the systems are shown to be equivalent. The same situation results in the case of cones (weak triples) and paths (weak cotriples) when the functors are adjoint. The relationship of adjointness to M - homotopy is also investigated.

Chapter VI consists of a detailed description of homotopy systems in additive categories. It is shown that homotopy systems are equivalent to pre cones while natural homotopy systems are the same as cones. Assuming cones then the study of homotopy systems in additive categories is similar to the study of certain classes of objects, in particular contractible objects defined via the cones. For example in the Eckmann-Hilton injective homotopy theory for modules the contractible objects are the injective modules. These homotopy categories are also compared with quotient categories or categories of fractions.

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CHAPTER I

BASIC CATEGORY THEORY AND TERMINOLOGY

1.1 Introduction.

This chapter contains a brief introduction to category theory, including the definitions, terminology and some basic results needed in this thesis. Sections two through five contain standard material which can be found in any introductory book on category theory, in particular Mitchell [14]. The sixth section is an introduction to homotopy and fibrations in a very general sense and contains some elementary results on these two notions.

1.2 Categories - Special Morphisms and Objects.

A *category* $\mathcal{A}$ is a class of *objects* $A,B,\cdots,X,Y,\cdots$ denoted by $|\mathcal{A}|$ together with a disjoint family of sets $\mathrm{Hom}(A,B;\mathcal{A})$ or $H(A,B)$ one for each pair of objects. The elements of $H(A,B)$ are called *morphisms* from A to B and for $f \in H(A,B)$ we write $f : A \longrightarrow B$. Furthermore for each $A,B,C \in |\mathcal{A}|$ and morphisms $f \in H(B,C)$ and $g \in H(A,B)$ there exists a uniquely defined product (composition) $f \circ g$ or $fg \in H(A,C)$ and this product has two properties:

(i) It is associative; if we have $h : A \longrightarrow B$, $g : B \longrightarrow C$ and $f : C \longrightarrow D$, then $(fg)h = f(gh)$.

(ii) For each $A \in |\mathcal{A}|$, there is a $1_A \in H(A,A)$ such that for each $f \in H(A,B)$, $1_B \circ f = f = f \circ 1_A$.

Certain morphisms in a category $\mathcal{A}$ are of special interest. A morphism f is said to be an *epimorphism* if $\alpha f = \beta f$ implies $\alpha = \beta$; f is a *monomorphism* if $f\alpha = f\beta$ implies $\alpha = \beta$. A morphism f is said to be a *retraction* (*coretraction*) if it is right (left) invertible, i.e., there exists a morphism f' $(f")$ such that $ff' = 1$ $(f"f = 1)$, where 1 is the identity morphism. A morphism which is both a retraction and a coretration is said to be invertible or an *isomorphism*. If $f \in H(A,B)$ is an isomorphism, the objects A and B are said to be *isomorphic*, $A \cong B$.

Certain objects are also of special interest. An object 0 in a category $\mathcal{A}$ is said to be a *null* (*conull*) *object* for $\mathcal{A}$ if $H(A,0)$ $(H(0,A))$ has precisely one element for each $A \in |\mathcal{A}|$. 0 is called a *zero object* for $\mathcal{A}$ if it is both a null and a conull object. In this case a morphism $A \longrightarrow B$ which factors through 0 is called a *zero morphism*.

We remark that $\mathcal{T}$, the category of topological spaces and continuous maps has a null object, the one-point space, a conull object, the empty space, but no zero object. Thus we usually restrict ourselves to $\mathcal{T}_*$, the category of topological spaces with base point and base point preserving continuous maps, for there the one-point space is a zero object.

1.3 Some Constructions in Categories.

Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be morphisms in a category $\mathcal{A}$. An object Q together with morphisms $f' : C \longrightarrow Q$ and $g' : B \longrightarrow Q$.

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d, "f'"'] \arrow[ddr, "f''"] & \\
B \arrow[r, "g'"] \arrow[drr, "g''"'] & Q \arrow[dr, dashed, "\gamma"] & \\
& & Q'
\end{tikzcd}$$

is called a *pushout* for f and g if $f'g = g'f$ and if $f'' : C \longrightarrow Q'$ and $g'' : B \longrightarrow Q'$ is another pair of morphisms to an object Q' with $f''g = g''f$ then there exists a unique morphism $\gamma : Q \longrightarrow Q'$ with $\gamma f' = f''$ and $\gamma g' = g''$.

Let $f : B \longrightarrow A$ and $g : C \longrightarrow A$ be morphisms in a category $\mathcal{A}$. An object P together with morphisms $f' : P \longrightarrow C$ and $g' : P \longrightarrow B$

$$\begin{tikzcd}
P' \arrow[dr, dashed, "\gamma"] \arrow[drr, "g''"] \arrow[ddr, "f''"'] & & \\
& P \arrow[r, "g'"] \arrow[d, "f'"] & B \arrow[d, "f"] \\
& C \arrow[r, "g"'] & A
\end{tikzcd}$$

is called a *pullback* for f and g if $fg' = gf'$ and if $f'' : P' \longrightarrow C$ and $g'' : P' \longrightarrow B$ is another pair of morphisms from an object P' with $fg'' = gf''$ then there is a unique morphism $\gamma : P' \longrightarrow P$ with $g'\gamma = g''$ and $f'\gamma = f''$.

Pushouts and pullbacks are unique up to isomorphism if they exist. We say that a *category has pushouts* if the pushout of every pair of morphisms with the same domain exists. This applies similarly for pullbacks.

Let $f : A \longrightarrow B$ be a morphism in a category with null object and pushouts. Then the *cokernel* of f , $B/f(A)$, is defined by the following pushout diagram:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & B/f(A) \end{tikzcd} .$$

Similarly if $\mathcal{A}$ has a conull object and pullbacks we define the *kernel* of a morphism as a pullback.

Let $\{A_i\}_{i\in I}$ be a family of objects in an arbitrary category $\mathcal{A}$. A *product* for this family is an object A together with a family of morphisms $\{p_i : A \longrightarrow A_i\}_{i\in I}$ with the property that for any other family $\{f_i : A' \longrightarrow A_i\}_{i\in I}$ there is a unique morphism $f : A' \longrightarrow A$ such that $p_i \circ f = f_i$ for all $i \in I$. We call p_i the projection and denote the product, A , by $\prod_{i\in I} A_i$ and the unique f into the product by $\{f_i\}_{i\in I}$. If I is a finite set we denote f by $\{f_1, f_2, \ldots, f_n\}$.

Similarly the *coproduct* (*sum*) of a family $\{A_i\}_{i\in I}$ of objects in a category $\mathcal{A}$ is an object A together with a family $\{j_i : A_i \longrightarrow A\}_{i\in I}$ with the property that for any other family $\{f_i : A_i \longrightarrow A'\}_{i\in I}$ there is a unique morphism $f : A \longrightarrow A'$ such that $f \circ j_i = f_i$ for all $i \in I$. We call j_i the injection and denote the coproduct, A , by $\bigoplus_{i\in I} A_i$ and the unique f out of the coproduct by $\langle f_i\rangle_{i\in I}$.

If I is a finite set we denote f by $\langle f_1, f_2, \ldots, f_n \rangle$.

1.4 Additive Categories - Injectivity.

An arbitrary category A is said to be *additive* if

(i) $H(A,B)$ is an abelian group, for all $A,B \in |A|$.

(ii) The composition of morphisms is bilinear.

(iii) There exists a zero object.

(iv) Finite products and coproducts exist in A .

In an additive category every finite product (coproduct) is a coproduct (product).

An additive category is said to be *abelian* if every morphism has a kernel and a cokernel, every monomorphism is the kernel of some morphism and every epimorphism is the cokernel of some morphism

An object Q in a category A is called *injective* if for any diagram

$$\begin{tikzcd} & Q \\ A \arrow[ur] \arrow[r] & B \arrow[u, dashed] \end{tikzcd}$$

where $A \longrightarrow B$ is a monomorphism, there is a morphism $B \longrightarrow Q$ making the diagram commutative.

We say that a *category* $\mathcal{A}$ *has injectives* if for every object A in $\mathcal{A}$ there is an injective object Q in $\mathcal{A}$ and a monomorphism $A \longrightarrow Q$.

1.5 Functors - Naturality, Equivalence.

Let $\mathcal{A},\mathcal{B}$ be two categories. A *covariant functor* $T : \mathcal{A} \longrightarrow \mathcal{B}$ is an assignment of an object $T(A)$ in $\mathcal{B}$ to each object A in $\mathcal{A}$, and a morphism $T(f) : T(A) \longrightarrow T(B)$ in $\mathcal{B}$ to each morphism $f : A \longrightarrow B$ in $\mathcal{A}$, subject to the following conditions:

(i) $T(1_A) = 1_{T(A)}$, for each $A \in |\mathcal{A}|$,

(ii) $T(f \circ g) = T(f) \circ T(g)$, whenever the morphism $f \circ g$ is defined in $\mathcal{A}$.

If in the above to each morphism $f : A \longrightarrow B$ in $\mathcal{A}$ there is assigned a morphism $T(f) : T(B) \longrightarrow T(A)$ in $\mathcal{B}$ and (ii) is replaced by (ii)'; $T(f \circ g) = T(g) \circ T(f)$, whenever the morphism $f \circ g$ is defined in $\mathcal{A}$, then T is termed a *contravariant functor.*

Let $S,T : \mathcal{A} \longrightarrow \mathcal{B}$ be covariant functors from a category $\mathcal{A}$ to a category $\mathcal{B}$. Suppose that for each object A in $\mathcal{A}$ we have a morphism $\eta(A) : S(A) \longrightarrow T(A)$ in $\mathcal{B}$ such that for every morphism $f : A \longrightarrow B$ in $\mathcal{A}$, the diagram

$$\begin{tikzcd} S(A) \arrow[r, "\eta(A)"] \arrow[d, "S(f)"'] & T(A) \arrow[d, "T(f)"] \\ S(B) \arrow[r, "\eta(B)"'] & T(B) \end{tikzcd}$$

is commutative. Then we call η a *natural transformation* from S to T and we write $\eta : S \longrightarrow T$. If $\eta(A)$ is an isomorphism in $\mathcal{B}$ for each $A \in |\mathcal{A}|$ then η is called a *natural equivalence*. In this case we nave a natural equivalence $\eta^{-1} : T \longrightarrow S$ defined by $(\eta^{-1})(A) = (\eta(A))^{-1}$.

If $\eta : S \longrightarrow T$ and $\rho : T \longrightarrow U$ are natural transformations of functors for $S,T,U : \mathcal{A} \longrightarrow \mathcal{B}$ covariant functors, then we have a composition $\rho\eta : S \longrightarrow U$ defined by $(\rho\eta)(A) = \rho(A) \circ \eta(A)$. For any functor T we have the identity transformation $1_T : T \longrightarrow T$ given by $1_T(A) = 1_{T(A)}$ for all $A \in |\mathcal{A}|$. If $S,T : \mathcal{A} \longrightarrow \mathcal{B}$, $U : \mathcal{B} \longrightarrow \mathcal{C}$ and $\eta : S \longrightarrow T$, then we have a natural transformation $U\eta : US \longrightarrow UT$ defined by $(U\eta)(A) = U(\eta(A))$ for all $A \in |\mathcal{A}|$. Similarly if $V : \mathcal{D} \longrightarrow \mathcal{A}$, then $\eta V : SV \longrightarrow TV$

is given by $(\eta V)(D) = \eta(V(D))$, for all $D \in |\mathcal{D}|$.

A functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ is called an *isomorphism* between the categories $\mathcal{A}$ and $\mathcal{B}$ if there exists a functor $T : \mathcal{B} \longrightarrow \mathcal{A}$ with $ST = 1_{\mathcal{B}}$ and $TS = 1_{A}$. In this case the categories $\mathcal{A}$ and $\mathcal{B}$ are said to be isomorphic.

A functor $S : \mathcal{A} \longrightarrow \mathcal{B}$ is called an *equivalence* between the categories $\mathcal{A}$ and $\mathcal{B}$ if there exists a functor $T : \mathcal{B} \longrightarrow \mathcal{A}$ and natural transformations $\eta : 1_{A} \longrightarrow TS$ and $\rho : 1_{\mathcal{B}} \longrightarrow ST$. In this case the categories $\mathcal{A}$ and $\mathcal{B}$ are said to be equivalent.

1.6 Homotopy and Fibrations.

In this section we give an approach to homotopy and fibrations based solely on "natural" equivalence relations. Although this is much too general to take as the "proper" approach to the study of homotopy and fibrations in arbitrary categories, it does have some interesting properties and gives a good introduction to the subject.

If $\mathcal{A}$ is any category, a *homotopy* on $\mathcal{A}$ is a natural equivalence relation, $\simeq$, on each of the sets, $H(A,B)$, of morphisms between the objects of $\mathcal{A}$. This means that $\simeq$ is an equivalence relation which is compatible with composition, that is $f \simeq g$ implies $fh \simeq gh$ and $kf \simeq kg$ whenever fh

and kf are defined. For $f \simeq g$ we say that f is homotopic to g with respect to $\simeq$.

If a category $\mathcal{A}$ has a homotopy, $\simeq$, the *homotopy category* of $\mathcal{A}$ with respect to $\simeq$, $\mathcal{A}/\simeq$, is the category whose objects are those of $\mathcal{A}$ and whose morphisms are the equivalence classes of morphisms of $\mathcal{A}$ under $\simeq$. If f is a morphism of $\mathcal{A}$, the corresponding morphism in $\mathcal{A}/\simeq$ is denoted by $[f]$. There is a natural projection functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\simeq$ which is the identity on objects and sends a morphism to the equivalence class of that morphism, $\pi(f) = [f]$, for f a morphism in $\mathcal{A}$. π is a covariant functor.

Two objects A,B in $\mathcal{A}$ are said to be *homotopically equivalent* if they are isomorphic in the homotopy category. This means there are morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow A$ in $\mathcal{A}$ such that $gf \simeq 1_A$ and $fg \simeq 1_B$ or $\pi(gf) = 1_A$ and $\pi(fg) = 1_B$. We write $A \simeq B$.

<u>Proposition 1.1</u> (<u>Universal Property of the Homotopy Functor</u>): Let $\mathcal{A}$ be a category with homotopy $\simeq$ and natural projection functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\simeq$. Then if T is any other covariant functor to a category $\mathcal{B}$ with the property that $f \simeq g$ implies $T(f) = T(g)$ for morphisms f and g in $\mathcal{A}$, there is a unique covariant functor $\Delta : \mathcal{A}/\simeq \longrightarrow \mathcal{B}$ with $\Delta\pi = T$.

Any covariant functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ will determine a homotopy on $\mathcal{A}$ by defining $f \simeq g$ whenever $F(f) = F(g)$ and f and g have the same domain and range. In this case we write $f \simeq g(F)$. If F is a one-one correspondence on objects and onto on morphisms then clearly $\mathcal{B} \cong \mathcal{A}/\simeq$.

<u>Homotopy Lifting Property</u>. Let $\mathcal{A}$ be a category with homotopy, $\simeq$, and natural projection functor $\pi : \mathcal{A} \longrightarrow \mathcal{A}/\simeq$. A morphism $p : E \longrightarrow B$ in $\mathcal{A}$ is called a $\simeq$ - *fibration* if for all objects X in $\mathcal{A}$ and all morphisms $f, f' : X \longrightarrow B$, $g : X \longrightarrow E$ in $\mathcal{A}$ with $pg = f$ and $f \simeq f'$,

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "g"] \arrow[r, "f \simeq f'"'] & B \end{tikzcd}$$

there exists a morphism $g' : X \longrightarrow E$ in with $pg' = f'$ and $g' \simeq g$. Concerning these $\simeq$ - fibrations we have the following two propositions:

<u>Proposition 1.2</u>:

(i) Every isomorphism is a fibration.

(ii) The composition of fibrations is a fibration.

(iii) All trivial morphisms $p : E \longrightarrow *$ are fibrations.

<u>Proposition 1.3</u>: Let $p : E \longrightarrow B$ be a fibration. Then

(i) If there exists $s : B \longrightarrow E$ with $ps \simeq 1$, then there exists $s' : B \longrightarrow E$ with $ps' = 1$ and $s' \simeq s$.

(ii) If $pg \simeq fp'$ in the following diagram

$$\begin{tikzcd} E' \arrow[r, "g"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

then there exists a $g' : E' \longrightarrow E$ with $pg' = fp'$ and $g' \simeq g$.

(iii) If there exists a commutative diagram

$$\begin{tikzcd} E' \arrow[r, "\alpha"] \arrow[d, "p'"] & E \arrow[r, "\beta"] \arrow[d, "p"] & E' \arrow[d, "p'"] \\ B' \arrow[r, "\mu"] & B \arrow[r, "\nu"] & B' \end{tikzcd}$$

with $\beta\alpha \simeq 1$ and $\nu\mu = 1$, then p' is a fibration.

The proof of these two propositions follows by an application of the definition.

CHAPTER II

HOMOTOPY SYSTEMS IN CATEGORIES

2.1 Introduction.

The method of considering homotopy and fibrations as being determined by a natural equivalence relation in a category is obviously very general and does not yield many interesting results. Thus we investigate specific methods of obtaining natural equivalence relations which will be of value and which will have as examples the usual homotopy notions that we are familiar with besides giving more examples which will show the generalities and potential of these approaches. We first introduce homotopy systems as defined by Kan [13] and studied with relation to "topological categories" by Kamps [10], [11] and, [12]. We conclude this chapter by introducing cones and contractibility into these systems.

2.2 Homotopy Systems.

The definition and some of the examples given here may be found in Kan [13] and Kamps [10].

Definition 2.1: Let A be a category. A *homotopy system* in A is a quadruple, $z = (Z, i_o, i_1, q)$, where $Z : A \longrightarrow A$ is a covariant functor and $i_o, i_1 : 1_A \longrightarrow Z$ and $q : Z \longrightarrow 1_A$ are natural transformations of functors with $q \circ i_o = 1 = q \circ i_1$.

Z is called a *cylinder functor*.

Example 2.2: There is the trivial homotopy system in any category, $e = (1_A, 1, 1, 1)$.

Example 2.3: In an additive category A, there is the opposite trivial homotopy system, $\bar{e} = (1_A \oplus 1_A, \{1,0\}, \{1,1\}, \langle 1,0 \rangle)$.

Examples 2.4: Let T be the category of topological spaces and continuous maps. Let $Z : T \longrightarrow T$ be the covariant functor given by $Z(X) = X \times I$, where I is the closed unit interval $[0,1]$, and $Z(f) = f \times 1_I$ for a continuous map f. If $X \in |T|$, let $i_o(X)$, $i_1(X) : X \longrightarrow X \times I$ and $q(X) : X \times I \longrightarrow X$ be given by $i_o(X)(x) = (x,0)$, $i_1(X)(x) = (x,1)$ and $q(X)(x,t) = x$, for $x \in X$, $t \in I$. Then $q \circ i_o = 1 = q \circ i_1$ and we have the usual homotopy system in T, $t = (Z, i_o, i_1, q)$.

Example 2.5: Let $C(A)$ be the category of chain complexes of an abelian category A. We define a homotopy system,

(Z,i_o,i_1,q) , in $\mathcal{C}(A)$. This construction differs from that given by Kamps [10] but the resulting homotopy relations, see §2.8, are the same and this construction coincides with the general approach taken to homotopy systems in additive categories given in chapter VI. For $(X,\partial) \in |\mathcal{C}(A)|$, $\partial_n : X_n \longrightarrow X_{n-1}$, let $Z(X)_n = X_n \oplus X_n \oplus X_{n-1}$ and $\delta_n : Z(X)_n \longrightarrow Z(X)_{n-1}$ be given by

$$\delta_n = \begin{pmatrix} \partial_n & 0 & 0 \\ 0 & \partial_n & 1_{X_{n-1}} \\ 0 & 0 & -\partial_{n-1} \end{pmatrix} .$$

Then $(Z(X),\delta) \in |\mathcal{C}(A)|$. If $f : X \longrightarrow Y$ is a chain map in $\mathcal{C}(A)$, let $Z(f)$ be the chain map given by $Z(f)_n = f_n \oplus f_n \oplus f_{n-1}$. Let $i_o(X)$, $i_1(X) : X \longrightarrow Z(X)$ and $q(X) : Z(X) \longrightarrow X$ be given by $i_o(X)_n = \{1_{X_n},0,0\}$, $i_1(X)_n = \{1_{X_n},1_{X_n},0\}$ and $q(X)_n = \langle 1_{X_n},0,0\rangle$. It follows that $i_o(X)$, $i_1(X)$ and $q(X)$ are chain maps, that i_o , i_1 and q are natural transformations of functors with $q \circ i_o = 1 = q \circ i_1$. Therefore (Z,i_o,i_1,q) is a homotopy system in $\mathcal{C}(A)$.

<u>Example 2.6</u> <u>(See Brown [2])</u>: Let A be the category of groupoids, i.e. every object of A is a small category in which every morphism has an inverse. Let I be the groupoid having two objects 0,1 , and two non-identity morphisms $i : 0 \longrightarrow 1$ and

$i^{-1} : 1 \longrightarrow 0$. Let $Z : A \longrightarrow A$ be the covariant functor assigning $G \times I$ to the groupoid G in A and assigning $f \times 1_I$ to a morphism f in A . Then $i_o(G)$, $i_1(G) : G \longrightarrow G \times I$ and $q(G) : G \times I \longrightarrow G$ given by $i_o(G)(g) = (g,0)$, $i_1(G)(g) = (g,1)$ and $q(G)(g,\varepsilon) = g$, for $g \in G$, $\varepsilon \in I$, give natural transformations functors i_o , i_1 , and with $q \circ i_o = 1 = q \circ i_1$. Thus $z = (Z, i_o, i_1, q)$ is a homotopy system in A .

Example 2.7: Let M_R be the category of right-modules over the commutative ring R . In this category there is the classical Eckmann-Hilton injective homotopy theory developed in detail in Hilton [7]. We shall show that this homotopy actually comes from a homotopy system in M_R . This fact is very significant since it lends validity to the development of homotopy systems in categories and the algebraic ideas give insight into the structure of topological homotopy. This homotopy system does not appear elsewhere in the literature and so we devote all of chapter III to its development.

2.3 Homotopy Relations.

Definition 2.8: Let A be a category and $z = (Z, i_o, i_1, q)$ a homotopy system in A . Let $f,g : X \longrightarrow Y$ be morphisms in A . We say f is *homotopic* to g , $f \simeq g(z)$ or simply

$f \simeq g$, if there exists a morphism $\phi : Z(X) \longrightarrow Y$

$$\begin{tikzcd} Z(X) \arrow[rrrr, bend left, dashed, "\phi"] & & X \arrow[ll, shift right, "i_o(X)"'] \arrow[ll, shift left, "i_1(X)"] \arrow[rr, shift left, "f"] \arrow[rr, shift right, "g"'] & & Y \end{tikzcd}$$

such that $\phi \circ i_o(X) = f$ and $\phi \circ i_1(X) = g$.

Such a morphism ϕ is called a *homotopy* from f to g , $\phi : f \simeq g$.

<u>Example 2.9</u>: In example 2.2, $f \simeq g$ if and only if $f = g$ while in 2.3, $f \simeq g$ for all g,f with the same domain and range. In example 2.4, $f \simeq g$ if and only if f is homotopic to g in the usual topological sense. In 2.5, $f \simeq g$ if and only if f is chain homotopic to g . If in example 2.6 we take the category of groups as a subcategory of the category of groupoids, for $f,g : G \longrightarrow H$, $f \simeq g$ if and only if there is a y in H with $f(x) = y + g(x) - y$ for all x in G . We shall see that for the Eckmann-Hilton injective homotopy $f \simeq g$ if and only if $f-g$ factors through an injective object.

<u>Remark 2.10</u>: The relation, "$\simeq$" , is in general not an equivalence relation. However it is reflexive and has the property that if $f \simeq g$ and fk is defined, then $fk \simeq gk$,

and also if kf is defined, then $kf \simeq kg$. In the examples given it is an equivalence relation. Moreover, since it generates an equivalence relation, we shall henceforth assume that the homotopy relation determined by a homotopy system is a natural equivalence relation.

We can make some statements about the symmetric property of this relation as follows:

<u>Lemma 2.11</u> <u>(Eckmann-Hilton [4])</u>: The homotopy relation given by a homotopy system $z = (Z, i_o, i_1, q)$ in a category A is symmetric if and only if there is a natural transformation of functors $r : Z \longrightarrow Z$ such that $r \circ i_o = i_1$ and $r \circ i_1 = i_o$.

<u>Lemma 2.12</u>: In a category with pushouts, the pushout of a homotopy system is a homotopy system.

<u>Proof</u>: Let $z = (Z, i_o, i_1, q)$ be a homotopy system and consider the following pushout diagram

$$\begin{tikzcd}
X \arrow[r, "i_o(X)"] \arrow[d, "i_o(X)"'] & Z(X) \arrow[d, "\alpha(X)"'] \arrow[ddr, "q(X)"] & \\
Z(X) \arrow[r, "\beta(X)"] \arrow[drr, "q(X)"'] & Z'(X) \arrow[dr, dashed, "q'(X)"] & \\
 & & X
\end{tikzcd}$$

Then $z' = (Z', \beta\circ i_1, \alpha\circ i_1, q')$ is a homotopy system.

<u>Corollary 2.12 A</u>: If z' is the pushout of a homotopy system z' as given in the lemma, then for morphisms f and g, $f \simeq g(z)$ implies $f \simeq g(z')$.

<u>Proof</u>: Assume $f,g : X \longrightarrow Y$ and that ϕ is a homotopy between f and g with respect to z. Consider the pushout diagram

$$\begin{tikzcd}
X \arrow[r, "i_o(X)"] \arrow[d, "i_o(X)"'] & Z(X) \arrow[d, "\alpha(X)"'] \arrow[ddr, "\phi"] & \\
Z(X) \arrow[r, "\beta(X)"] \arrow[drr, "\phi\circ i_o(X)\circ q(X)"'] & Z'(X) \arrow[dr, dashed, "\phi'"] & \\
& & Y
\end{tikzcd}$$

We have $\phi \circ i_o(X) \circ q(X) \circ i_o(X) = \phi\circ i_o(X)$. Hence there is a $\phi' : Z'(X) \longrightarrow Y$ with $\phi'\circ\alpha(X) = \phi$ and $\phi'\circ\beta(X) = \phi\circ i_o(X)$ $\circ q(X)$. Clearly $\phi'\circ\beta(X) \circ i_1(X) = \phi\circ i_o(X) \circ q(X) \circ i_o(X) =$ $\phi\circ i_o(X) = f$ and $\phi'\circ \alpha(X) \circ i_1(X) = \phi \circ i_1(X) = g$. Thus $\phi' : Z'(X) \longrightarrow Y$ is a homotopy between f and g with respect to the homotopy system z'.

It also follows that $\simeq (z')$ is symmetric, thus allowing us to introduce symmetry into our homotopy relation. This comes from the existence of $r : Z'(x) \longrightarrow Z'(X)$ satisfying $r \circ \alpha = \beta$ and $r\circ\beta = \alpha$, which follows from the commutativity of

$$\begin{tikzcd}
X \arrow[r, "i_o"] \arrow[d, "i_o"'] & Z(X) \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & \\
Z(X) \arrow[r, "\beta"] \arrow[drr, "\alpha"'] & Z'(X) \arrow[dr, dashed, "r"] & \\
 & & Z'(X)
\end{tikzcd}$$

2.4 <u>Fibrations</u>.

Throughout this section we let A be an arbitrary category and $z = (z, i_o, i_1, q)$ be a homotopy system in A .

<u>Definition 2.13</u> <u>(Kamps [10])</u>. A morphism $P : E \longrightarrow B$ in A is called a z - *fibration* if any commutative diagram of the form

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "i_o(X)"'] & E \arrow[d, "p"] \\
Z(X) \arrow[r, "\phi"] \arrow[ur, dashed, "\overline{\phi}"] & B
\end{tikzcd}$$

can be filled in with a homotopy $\overline{\phi} : Z(X) \longrightarrow E$ such that

(i) $P\,\overline{\phi} = \phi$, and

(ii) $\overline{\phi} \circ i_o(X) = g$.

If (ii) is replaced by the following; (ii)' there exists a homotopy $\bar{\bar{\phi}} : Z(X) \longrightarrow E$ such that $\bar{\bar{\phi}} : g \simeq \bar{\phi} \circ i_o(X)$ and $p \circ \bar{\bar{\phi}} = p \circ g \circ q(X)$, then p is called an *h - fibration.*

Example 2.14: In example 2.2 every morphism is a z - fibration while in 2.3 the retractions are the z - fibrations. In the topological case, example 2.4, the z - fibrations are the Hurewicz fibrations while the h - fibrations are the Dold or weak fibrations. In example 2.6, the category of groupoids, the z - fibrations are the star surjective maps while if we take only the subcategory of groups the z - fibrations are the surjective maps.

From remark 2.10, we may assume that the homotopy relation, $\simeq (z)$, determined by the homotopy system z is a natural equivalence relation and so by 1.6 we may consider the $\simeq (z)$ - fibrations or simply $\simeq$ - fibrations, if no confusion arrises. All of the properties of $\simeq$ - fibrations given in propositions 1.2 and 1.3 hold for z - fibrations (the requirement in 1.3 (c) that $\beta\alpha \simeq 1$ has to be strenthened to $\beta\alpha = 1$). In addition we have

Proposition 2.15:

(a) Projection maps are z - fibrations.

(b) the pullback of a z - fibration is a z - fibration.

<u>Lemma 2.16</u> <u>(Kamps [10])</u>: Every z - fibration is an h - fibration and every h - fibration is a $\simeq$ (z) - fibration.

<u>Proof</u>: Clearly from the definition every z - fibration is an h - fibration. Thus let $p : E \longrightarrow B$ be an h - fibration and assume we have

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

$\phi : pg \simeq f$, $\phi \circ i_o(X) = pg$ and $\phi \circ i_1(X) = f$. This gives a commutation diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i_o(X)"'] & E \arrow[d, "p"] \\ Z(X) \arrow[ur, dashed, "\bar{\phi}"] \arrow[r, "\phi"] & B \end{tikzcd}$$

and p an h - fibration gives a $\bar{\phi} : Z(X) \longrightarrow E$ with $p\,\bar{\phi} = \phi$ and a $\bar{\bar{\phi}} : Z(X) \longrightarrow E$ with $\bar{\bar{\phi}} : \bar{\phi} \circ i_o(X) \simeq g$ and $p \circ \bar{\bar{\phi}} = p \circ g \circ q(X)$. Let $q' : X \longrightarrow E$ be given by $g' = \bar{\phi} \circ i_1(X)$. Then $pg' = p \circ \bar{\phi} \circ i_1(X) = \phi \circ i_1(X) = f$ and $g' = \bar{\phi} \circ i_1(X) \simeq \bar{\phi} \circ i_o(X) \simeq g$.

<u>Definition 2.17</u>: A morphism $j : A \longrightarrow X$ in $\mathcal{A}$ is called a *z - cofibration* if for any commutation diagram of the form

$$\begin{tikzcd}
A \arrow[r, "i_o(A)"] \arrow[d, "j"'] & Z(A) \arrow[d, "Z(j)"'] \arrow[ddr, "\phi"] & \\
X \arrow[r, "i_o(X)"] \arrow[drr, "f"'] & Z(X) \arrow[dr, dashed, "\overline{\phi}"] & \\
 & & Y
\end{tikzcd}$$

there exists a $\overline{\phi} : Z(X) \longrightarrow Y$ with $\overline{\phi} \circ i_o(X) = f$ and $\overline{\phi} \circ Z(j) = \phi$.

The inner square is called a *weak cocartesian square* or a *weak pushout* because $\overline{\phi}$ is not required to be unique.

2.5 <u>Cones and Contractibility</u>.

We end this chapter with a version of cones and contractibility which generalizes the topological notion and which pertains to the other examples given in §2.2. Some of these ideas may be found in Seebach [17].

<u>Definition 2.18</u>: A *pre cone* in a category $\mathcal{A}$ is a pair (C,i) where $C : \mathcal{A} \longrightarrow \mathcal{A}$ is a covariant functor and $i : 1 \longrightarrow C$ is a natural transformation of functors.

Every homotopy system $z = (Z,i_o,i_1,q)$ in a category $\mathcal{A}$ with a null object and pushouts gives a pre cone in $\mathcal{A}$ via the following pushout diagram

$$\begin{tikzcd} X \arrow[r, "i_o(X)"] \arrow[d] & Z(X) \arrow[d, "g(X)"] \\ * \arrow[r] & C(X) \end{tikzcd}$$

with $i(X) = g(X) \circ i_1(X)$. Then a morphism $f : X \longrightarrow Y$ in A is said to be *null homotopic* if and only if there is a morphism $F : C(X) \longrightarrow Y$ with $f = F \circ i(X)$.

Definition 2.19: An object X is *contractible* if the identity map of X is homotopic to a null morphism.

Thus for an object X in A, the cone of X, $C(X)$, is contractible if and only if there is a morphism $p : C^2(X) \longrightarrow C(X)$ with $p \circ i(C(X)) = 1_{C(X)}$. With the usual homotopy systems given in §2.2 and their resulting pre-cones, the cone of every object is contractible.

Definition 2.20: A pre cone (C,i) is a *cone* (C,i,p) if there is a natural transformation of functors $p : C^2 \longrightarrow C$ with $p \circ i(C) = 1_C$.

All the usual pre cones are cones and we shall see in chapter IV that every pair of adjoint functors and every triple yields a cone.

<u>Lemma 2.21</u>: If $(C,i) = (C,i,p)$, then a morphism is null homotopic if and only if it factors through the cone of some object.

<u>Proof</u>: If a morphism $f : X \longrightarrow Y$ is null homotopic then it factors through $i(X) : X \longrightarrow C(X)$. Conversely assume a morphism $f : X \longrightarrow Y$ factors as $f = \alpha \circ \beta$ for $\beta : X \longrightarrow C(A)$ and $\alpha : C(A) \longrightarrow Y$ for some object A .
We may form the following diagram:

$$\begin{tikzcd}
 & C(A) \arrow[rr, shift left, "i(C(A))"] \arrow[ddrr, "\alpha"] & & C^2(A) \arrow[ll, shift left, "p(A)"] \\
 & & C(X) \arrow[ur, "C(\beta)"'] \arrow[dr, dashed, "F"'] & \\
X \arrow[uur, "\beta"] \arrow[urr, "i(X)"'] \arrow[rrr, "f"'] & & & Y
\end{tikzcd}$$

Define $F : C(X) \longrightarrow Y$ by $F = \alpha \circ p(A) \circ C(\beta)$. Then
$F \circ i(X) = \alpha \circ p(A) \circ C(\beta) \circ i(X) = \alpha \circ p(A) \circ i(C(A)) \circ \beta =$
$\alpha \circ \beta = f$.

CHAPTER III

ECKMANN-HILTON INJECTIVE HOMOTOPY

3.1 Introduction.

The aim of this chapter is to develop in detail the classical Eckmann-Hilton injective homotopy theory for modules in terms of a homotopy system. The main idea in the construction is to embed an arbitrary module "functorially" into an injective module. The approach in based on that of Huber [8] and [9] in which he studies this and other homotopies in terms of the homotopy of semi-simplical complexes and Kan complexes.

3.2 Injective Homotopy System.

Let $\mathcal{A}$ be an abelian category with infinite direct products and let $\mathcal{M}$ be the category of sets. We define a contravariant functor $P : \mathcal{M} \longrightarrow \mathcal{A}$ as follows: For a fixed object U in $\mathcal{A}$, set

$$P(M) = \prod_{m \in M} U_m \quad , \quad U_m = U \; , \; \forall m \; , \; M \in |\mathcal{M}| \; ,$$

and if $v : M \longrightarrow M'$ is a morphism in $\mathcal{M}$, let

$P(v) : P(M') \longrightarrow P(M)$ be given coordinatewise by

$$P(v)_m = 1_U \circ p_{v(m)} \quad ,$$

where $p_{v(m)}$ is the projection map.

We define another contravariant functor $G : \mathcal{A} \longrightarrow \mathcal{M}$ as follows: For X an object in $\mathcal{A}$, set $G(X) = \mathrm{Hom}(X,U;\mathcal{A})$ and for $f : X \longrightarrow Y$ a morphism in $\mathcal{A}$, define $G(f) : G(Y) \longrightarrow G(X)$ by

$$G(f)(\alpha) = \alpha \circ f \quad , \quad \alpha \in \mathrm{Hom}(X,U;\mathcal{A}) \quad .$$

On the basis of the above construction we obtain

<u>Lemma 3.1</u>: P and G are adjoint on the right.

<u>Proof</u>: For $X \in |\mathcal{A}|$, $M \in |\mathcal{M}|$ we define a set function

$$\eta : \mathrm{Hom}(X, \prod_{m \in M} U_m;\mathcal{A}) \longrightarrow \mathrm{Hom}(M,\mathrm{Hom}(X,U;\mathcal{A});\mathcal{M})$$

by $\eta(\alpha)(m) = \alpha_m$. The inverse of η, η^{-1}, is given by $\eta^{-1}(\beta)_m = \beta(m)$.

Then $C = P \circ G$ is a covariant functor $\mathcal{A} \longrightarrow \mathcal{A}$. The adjoint situation gives a natural transformation of functors $k : 1_{\mathcal{A}} \longrightarrow C$ by

$$k(X) = \eta^{-1}(1_{G(X)}) \quad , \quad X \in |A| \quad .$$

The following definition and some of the ideas may be found in Mitchell [14], pages 73-74.

Definition 3.2: An object U in a category A is called a *cogenerator* for the category if whenever we have morphisms $f,g : X \longrightarrow Y$, $f \neq g$, there is a morphism $\alpha : Y \longrightarrow U$ with $\alpha f \neq \alpha g$.

Using the above definition and previous construction we obtain the following.

Proposition 3.3: If A has infinite products, then U is a cogenerator for A if and only if $k(X)$ is a monomorphism for all objects X in A .

Proof: Suppose that U is a cogenerative and consider

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{k(Y)} C(Y) = \prod_{m \in Hom(Y,U;A)} U_m$$

with $k(Y) \circ f = k(Y) \circ g$. Then $K(Y)_m \circ g = k(Y)_m \circ f$ for each m and so $m \circ f = m \circ g$, $\forall m$. If $f \neq g$, then since U is a cogenerator there exists an $m \in Hom(Y,U;A)$ with $m \circ f \neq m \circ g$. Thus $f = g$, and $k(Y)$ is a monomorphism.

Conversely assume $k(X)$ is a monomorphism for each object X in A. Let $f,g : X \longrightarrow Y$ be distinct morphisms. Then for some projection, P_m, from the product we must have

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{k(Y)} \Pi U_m \xrightarrow{P_m} U_m = U$$

$P_m \circ k(Y) \circ f = P_m \circ k(Y) \circ g$. Thus U is a cogenerator.

Proposition 3.4: If U is injective, then $C(X)$ is injective for all objects X in A.

Proof: The product of injectives is injective.

Remark 3.5: If a category has injectives and a cogenerator then it has an injective cogenerator. In particular if $f,g : X \longrightarrow Y$ and $f \neq g$, there is a cogenerator U and a morphism $\alpha : Y \longrightarrow U$ with $\alpha f \neq \alpha g$. Then there is also an injective Q and a monomorphism $j : U \longrightarrow Q$ with $j(\alpha f) \neq j(\alpha g)$. Thus $(j\alpha)f \neq (j\alpha)g$ and so Q is also a cogenerator.

Remark 3.6: The category of right R - modules, M_R, has an injective cogenerator, namely $\mathrm{Hom}(R,Q/Z;G)$, where G is the category of groups and Q/Z is the group of rationals modulo the integers.

We summarize the above results in

<u>Theorem 3.7</u>: Let A be an abelian category with infinite direct products and an injective cogenerator. Then there exists a covariant fuctor $C : A \longrightarrow A$ and a natural transformation of functors $k : 1 \longrightarrow C$

(a) $X \in |A|$ implies $C(X)$ is injective

(b) $k(X) : X \longrightarrow C(X)$ is a monomorphism for each object X in A .

3.3 <u>Homotopy and Fibrations</u>.

We proceed to use the results of the previous section to show that we have produced a homotopy system and that the resulting homotopy relation, in the category of modules, coincides with the classical Eckmann-Hilton injective homotopy relation and that the same holds for fibrations. For this section let A be an abelian category with infinite direct products and injective cogenerator.

We define a homotopy system, $z = (Z, i_o, i_1, q)$ in A as follows: Define $Z : A \longrightarrow A$ by $Z(X) = X \oplus C(X)$ and $Z(f) = f \oplus C(f)$, for X an abject in A and f a morphism in A . Define $i_o, i_1 : 1 \longrightarrow Z$ and $q : Z \longrightarrow 1$ by

$$i_o(X) = \{1_X, 0\} \ , \ i_1(X) = \{1_X, k(X)\} \quad \text{and} \quad q(X) = \langle 1_X, 0\rangle \ .$$

It follows that i_o, i_1, q are natural transformations of functors and that $q \circ i_o = 1 = q \circ i_1$. Thus $z = (Z, i_o, i_1, q)$ is a homotopy system in A .

<u>Theorem 3.8</u>: For $f, g : X \longrightarrow Y$ morphisms in A , $f \simeq g(z)$ it and only if $f\text{-}g$ factors through some injective object.

<u>Proof</u>: Assume $f\text{-}g$ factors through some injective object. Then $f\text{-}g$ factors through every injective object containing X . Thus there is a morphism $\pi : C(X) \longrightarrow Y$ with $f\text{-}g = \pi \circ k(X)$.

$$\begin{tikzcd} & C(X) \arrow[dr, "\pi"] & \\ X \arrow[ur, "k(X)"] \arrow[rr, "f-g"'] & & Y \end{tikzcd}$$

Define a homotopy $F : Z(X) \longrightarrow Y$ by $F = \langle f, -\pi\rangle$. Then $F \circ i_o(X) = \langle f, -\pi\rangle \circ \{1_X, 0\} = f$ and $F \circ i_1(X) = \langle f, -\pi\rangle \circ \{1_X, f(X)\} = f - \pi \circ k(X) = f - (f\text{-}g) = g$. Thus $F : f \simeq g$.

Conversely assume $f \simeq g$. Thus there is a homotopy $F = \langle F_1, F_2\rangle : X \oplus C(X) \longrightarrow Y$, with $F \circ i_o(X) = f$ and

$F \circ i_1(X) = g$. Then $f-g = F \circ i_o(X) - F \circ i_1(X) = \langle F_1, F_2 \rangle \circ \{1_X, 0\} - \langle F_1, F_2 \rangle \circ \{1_X, k(X)\} = F_1 - (F_1 + F_1 \circ k(X)) = - F_2 \circ k(X)$. Thus $f-g$ factors through $k(X) : X \longrightarrow C(X)$, and so $f-g$ factors through an injective object.

<u>Definition 3.9</u> <u>(Hilton [7])</u>: A morphism $P : E \longrightarrow B$ in $\mathcal{A}$ is an *injective fibration* if and only if for all injective objects Q in $\mathcal{A}$ every morphism $\beta : Q \longrightarrow B$ can be lifted to E .

$$\begin{tikzcd} & E \arrow[d, "p"] \\ Q \arrow[ur, dashed, "\alpha"] \arrow[r, "\beta"'] & B \end{tikzcd}$$

<u>Remark 3.10</u>: The injective homotopy relation or $\simeq (z)$ is, without extension a natural equivalence relation. In §1.6 we introduced the homotopy lifting property and the $\simeq (z)$ - fibrations. In the present situation we now have:

<u>Lemma 3.11</u>: A morphism $p : E \longrightarrow B$ in $\mathcal{A}$ is an injective fibration if and only if it is a $\simeq (z)$ - fibration.

The proof of this lemma is contained in the proof of a more general theorem for additive categories, theorem 6.10.

<u>Theorem 3.12</u>: A morphism $p : E \longrightarrow B$ in A is an injective fibration if and only if it is a z - fibration.

<u>Proof</u>: Let p be an injective fibration and consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\{1_X, 0\}"'] & E \arrow[d, "p"] \\ X \oplus C(X) \arrow[r, "<\beta_1, \beta_2>"'] & B \end{tikzcd}$$

Since p is an injective fibration, there exists a morphism $\gamma : C(X) \longrightarrow E$ with $p\gamma = \beta_2$. Define $F : X \oplus C(X) \longrightarrow E$ by $F = <\alpha, \gamma>$. Then $p \circ F = <\beta_1, \beta_2>$ and $F \circ \{1_X, 0\} = \alpha$. So p is a z - fibration.

The converse follows from lemma 3.11 and lemma 2.16.

CHAPTER IV

CATEGORIES OF FRACTIONS AND HOMOTOPY THEORY

4.1 Introduction.

Gabriel-Zisman [6] and Bauer-Dugundji [1] defined for each class of morphisms M in a category A, a category of fractions of A by M or a quotient category, A/M. The natural projection functor $\eta : A \longrightarrow A/M$ determines a natural equivalence relation, or a homotopy by §1.6, among the morphisms in A. Bauer-Dugundji [1] also defined for each class of morphisms M a fibration notion, M - fibration. We study in this chapter some of the properties of M - homotopy and M - fibrations and show how they are related to the homotopy systems of chapter II.

4.2 M - Homotopy.

Let A be any category, and let M be any family of its morphisms. By a *quotient category* we mean a pair, $(A/M, \eta)$, where A/M is a category with the same objects as A and $\eta : A \longrightarrow A/M$ is a covariant functor preserving objects and having the following two properties:

(i) If $\alpha \in M$, then $\eta(\alpha)$ is an isomorphism in A/M.

(ii) Universal property. If $T : A \longrightarrow B$ is any other covariant functor to a category B such that $T(\alpha)$ is an isomorphism for each $\alpha \in M$, then there is a unique covariant functor $\Delta : A/M \longrightarrow B$ with $\Delta \circ \eta = T$.

Theorem 4.1 (Bauer-Dugundji [1]): Let A be any category and let M be any family of its morphisms. Then a quotient category, $(A/M, \eta)$, exists.

Let M be any class of morphisms in an arbitrary category A, and let $\eta : A \longrightarrow A/M$ be the natural projection functor as above. Then by §1.6, we have a natural equivalence relation among the morphisms in A; if f,g are two morphisms in A, we write $f \simeq g(M)$ if $\eta(f) = \eta(g)$, and denote the corresponding homotopy category by $A/\simeq(M)$. This M - homotopy has the further property that if for two morphisms f,g in A, there is an $\alpha \in M$ with $\alpha f = \alpha g$ or $f\alpha = g\alpha$, then $f \simeq g(M)$.

Remark 4.2: If $\alpha \in M$ and α has a left (right) inverse β, then we may assume that β is also in M, because by the above $\beta\alpha = 1$ and $(\alpha\beta)\alpha = \alpha$ and so $\eta(\alpha) \circ \eta(\beta) = \eta(\alpha\beta) = \eta(1)$. So $\eta(\beta)$ is also an isomorphism in A/M with inverse $\eta(\alpha)$.

We may sometimes associate the resulting homotopy category with the quotient category by the following lemma:

<u>Lemma 4.3</u>: If A is any category and M a class of its morphisms with M a subset of the union of the retractions and the coretractions in A , then the quotient category, A/M , is isomorphic to the homotopy category, $A/\simeq(M)$.

<u>Proof</u>: Assume we have $\eta : A \longrightarrow A/M$ and $\pi : A \longrightarrow A/\simeq(M)$ with $\pi(f) = [f]$, the natural projection functors.

$$\begin{tikzcd} & A/M \arrow[d, shift right, dashed, "\Delta"'] \\ A \arrow[ur, "\eta"] \arrow[dr, "\pi"'] & \\ & A/\simeq(M) \arrow[uu, shift right, dashed, "\Gamma"'] \end{tikzcd}$$

If $\alpha : A \longrightarrow B$ is in M and has retract $\beta : B \longrightarrow A$ with $\beta\alpha = 1_A$, then by the above remarks $\eta(\alpha)$ is an isomorphism with inverse $\eta(\beta)$. Thus $\pi(\alpha) = [\alpha]$ is an isomorphism with inverse $[\beta]$ and by the universality of the quotient category, $(A/M, \eta)$, there is a unique functor $\Delta : A/M \longrightarrow A/\simeq(M)$ with $\Delta \circ \eta = \pi$. The same follows if α is a retract. Similarly if $f \simeq g(M)$, then $\pi(f) = \pi(g)$ and $\eta(f) = \eta(g)$ and by the universality of the homotopy category, proposition 1.1, there is a unique $\Gamma : A/\simeq(M) \longrightarrow A/M$ with $\Gamma \circ \pi = \eta$. Then from the universality of the quotient category and the homotopy category it follows that $\Delta\Gamma = 1$ and $\Gamma\Delta = 1$.

The following theorem of Bauer and Dugundji [1] tells us that by suitable choices of M, the M - homotopy notion will coincide with some usual homotopy notion. In particular it tells us when the homotopy category resulting from a homotopy system or cyclinder is a quotient category or a category of fractions.

Theorem 4.4: Let A be a cateogry and M a class of morphisms in A with $\eta : A \longrightarrow A/M$ the natural projection functor. Let "$\simeq$" be any natural equivalence relation among the morphisms of A and let $\pi : A \longrightarrow A/\simeq$ be the projection function to the homotopy category, $A/\simeq$. Suppose that if $f_o \simeq f_1 : X \longrightarrow Y$ there is an object Z_X in A and morphisms $j_o, j_1 ; X \longrightarrow Z_X$, $r : Z_X \longrightarrow X$ and $F : Z_X \longrightarrow Y$ with $r \circ j_\varepsilon = 1_X$ and $F \circ j_\varepsilon = f_\varepsilon$, $(\varepsilon=0,1)$. Then

(i) if $\eta(r)$ is an isomorphism in A/M, then $\pi(f_o) = \pi(f_1)$ implies $\eta(f_o) = \eta(f_1)$,

(ii) if $\pi(\alpha)$ is an isomorphism in $A/\simeq$ for each $\alpha \in M$, then $\eta(f_o) = \eta(f_1)$ implies $\pi(f_o) = \pi(f_1)$.

Definition 4.5: If for a homotopy system $z = (Z, i_o, i_1, q)$, $q(X) : Z(X) \longrightarrow X$ is a homotopy equivalence with homotopy inverse $i_o(X)$, we call z a *natural homotopy system.*

<u>Corollary 4.6</u>: Let $z = (Z, i_o, i_1, q)$ be a natural homotopy system in a category A. Let $M = \{i_o(X) : X \in |A|\}$. Then the homotopy category, $A/\simeq(z)$, determined by the homotopy system z and the homotopy category, $A/\simeq(M)$, determined by the class of morphisms M are both the same and are both the category of fractions of A by M, A/M.

This is the case for the homotopy systems given as examples in chapter II. In particular in the category of topological spaces we have X and $X \times I$ homotopically equivalent with $i_o(X)$ and $q(X)$ as the homotopy equivalences. It seems that there should be some relation in arbitrary categories between a homotopy system as a natural homotopy system and when the resulting pre cone is actually a cone. This is because in the examples given the reason for the homotopy system being a natural homotopy system and the resulting pre cone being a cone are both the same. For example in the category of topological spaces both of these follow because there is a map $r : X \times I \times I \longrightarrow X \times I$, with $r(x,s,t) = (x,st)$, and we mention in chapter VI a theorem of Seebach [17] which compares the two ideas for abelian categories.

Theorem 4.4 can also be used to reclaim a homotopy from the knowledge of contractible ojects in a category as we shall show in chapter VI in the general case for additive categories and as the following theorem of Seeback [17] shows in the specific case for the category of C.W. complexes.

<u>Theorem 4.7</u>: In the category $\mathcal{A}$ of C.W. Complexes, let $\mathcal{M}$ be the class of all coretractions with a contractible cokernel. Then the resulting $\mathcal{M}$ - homotopy is the usual homotopy for $\mathcal{A}$.

4.3 <u>$\mathcal{M}$ - Fibrations</u>.

For each family of morphisms $\mathcal{M}$ in a category $\mathcal{A}$, Bauer and Dugundji [1] also define a very general concept of fibration as follows:

<u>Definition 4.8</u>: A morphism $p : E \longrightarrow B$ in $\mathcal{A}$ is called an $\mathcal{M}$ - *fibration* if for each diagram

$$\begin{tikzcd} & & E \arrow[d, "p"] \\ W \arrow[r, "\mu"'] & X \arrow[ur, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

in which $pg\mu = f\mu$ and $\mu \in \mathcal{M}$, there is a $g' : X \longrightarrow E$ in $\mathcal{A}$ with $g\mu = g'\mu$ and $pg' = f$.

These $\mathcal{M}$ - fibrations have all the usual properties required of fibrations as follows:

<u>Theorem 4.9</u>: Let M be any family of morphisms in a category A. Then we have

(i) All isomorphisms are M - fibrations.

(ii) The composition of two M - fibrations is an M - fibration.

(iii) The pullback of an M - fibration is an M - fibration.

(iv) The product of two M - fibrations is an M - fibration.

(v) All trivial morphisms, $p : E \longrightarrow *$, are M - fibrations.

<u>Remark 4.10</u>: We denote the class of all M - fibrations by $\{M\text{-Fib.}\}$, for each family of morphisms M in a category A. Clearly if $M \subset N$, then $\{N\text{-Fib.}\} \subset \{M\text{-Fib.}\}$. Also if M is any subset of the set of all retractions in A then $\{M\text{-Fib.}\}$ is the class of all morphisms in A.

If we have a homotopy system $z = (Z, i_o, i_1, q)$ in a category A, then the z - fibrations may be associated with the above classes of fibrations by

<u>Theorem 4.11</u>: Let $M = \{i_o(X) : X \in |A|\}$. Then $\{M\text{-Fib.}\}$ is the class of all z - fibrations.

<u>Proof</u>: Let $p : E \longrightarrow B$ be a z - fibration and consider the diagram

$$\begin{tikzcd} & & E \arrow[d, "p"] \\ X \arrow[r, "i_o(X)"'] & Z(X) \arrow[ur, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

with $p \circ g \circ i_o(X) = f \circ i_o(X)$. Since p is a z - fibration and $p \circ (g \circ i_o(X)) = f \circ i_o(X)$, there exists a homotopy $\phi : Z(X) \longrightarrow E$ with $\phi \circ i_o(X) = g \circ i_o(X)$ and $p \circ \phi = f$. Thus p is an M - fibration.

Conversely assume p is an M - fibration and that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "i_o(X)"'] & E \arrow[d, "p"] \\ X(Z) \arrow[r, "f"'] & B \end{tikzcd}$$

Now $p \circ (g \circ q(X)) \circ i_o(X) = f \circ i_o(X)$ and since p is an M - fibration there is a $g' : Z(X) \longrightarrow E$ with $p \circ g' = f$ and $g' \circ i_o(X) = (g \circ q(X)) \circ i_o(X) = g$. Thus p is a z - fibration.

Remark 4.12: If we consider additive categories and take the trivial homotopy system $\overline{e} = (1 \oplus 1 \, , \{1,0\} \, , \{1,1\} \, , \langle 1,0 \rangle)$ given in example 2.3, then the M from the above theorem is a subclass of L , the class of all coretractions. In §2.10 we said that the $\overline{e}$ - fibrations were the class of all retractions. Thus $\{L\text{-Fib.}\} \subset \{\text{retractions}\}$. Furthermore it can easily be seen that in additive categories a retraction is an M - fibration for every family of morphisms M . Thus in additive categories the retractions or $\{L\text{-Fib.}\}$ form the smallest class of $\{M\text{-Fib.}\}$.

CHAPTER V

HOMOTOPY, TRIPLES AND ADJOINTNESS

5.1 Introduction.

In this chapter we introduce cohomotopy systems and the resulting homotopy and fibration notions. We see that if the cylinder functor in a homotopy system has an adjoint then we may construct a cohomotopy system such that the two homotopy notions coincide, and also so do the two fibrations notions. We then investigate the relation of adjointness to the M - homotopy of chapter IV. Finally we briefly mention triples as being a strong form or a cone and remark that triples and adjointness are very similar.

5.2 Cohomotopy Systems.

Definition 5.1: A *cohomotopy system* in a category A is a quadruple $w = (W,p_o,p_1,k)$, where $W : A \longrightarrow A$ is a covariant functor and $p_o,p_1 : W \longrightarrow 1_A$ and $k : 1_A \longrightarrow W$ are natural transformations of functors with $p_o \circ k = 1 = p_k \circ k$.

Example 5.2: In every category A there is the trivial cohomotopy system $w = (1_A,1,1,1)$.

<u>Example 5.3</u>: In $\mathcal{T}$, the category of topological spaces and continuous maps we have the usual cohomotopy system $w = (W, p_o, p_1, k)$ given as follows: Let I be the closed unit interval $[0,1]$. Then for $X \in |\mathcal{T}|$, define $W(X) = X^I$, the set of continuous maps $I \longrightarrow X$ with the compact open topology, and for $f : X \longrightarrow Y$ a continuous map in $\mathcal{T}$, define $W(f) : X^I \longrightarrow Y^I$ by $W(f)(\omega)(t) = f(\omega(t))$, for $\omega \in X^I$ and $t \in I$. For $X \in |\mathcal{T}|$, define $p_o(X)$, $p_1(X) : X^I \longrightarrow X$ and $k(X) : X \longrightarrow X^I$ by $p_o(X)(\omega) = \omega(0)$, $p_1(X)(\omega) = \omega(1)$ and $k(X)(x) = C_x$, the constant path at x. Then $p_o \circ k = 1 = p_1 \circ k$ and so we have a cohomotopy system in $\mathcal{T}$.

<u>Example 5.4</u>: Dual to the Eckmann-Hilton injective homotopy theory for modules, there is a projective homotopy theory for modules. Here, two morphisms are (projectively) homotopic if and only if their difference factors through a projective module. In a similar way to that of chapter III this theory can be shown to be an example of a cohomotopy system in the category of modules.

<u>Definition 5.5</u>: Let $w = (W, p_o, p_1, k)$ be a cohomotopy system in a category $\mathcal{A}$. Then for morphisms $f, g : X \longrightarrow Y$ in $\mathcal{A}$, we say that f is *cohomotopic* to g, $f \simeq g(w)$, if there exists a morphism $\phi : X \longrightarrow W(Y)$

$$\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[rr, dashed, bend left=40, "\phi"] & Y & W(Y) \arrow[l, shift right, "p_o(W)"'] \arrow[l, shift left, "p_1(W)"] \end{tikzcd}$$

such that $p_o(Y) \circ \phi = f$ and $p_1(Y) \circ \phi = g$.

Definition 5.6: Let $w = (W,p_o,p_1,k)$ be a cohomotopy system in a cateogry A . A morphism $p : E \longrightarrow B$ in A is called a *w - fibration* if for any commutative diagram

$$\begin{tikzcd} X \arrow[drr, bend left, "f"] \arrow[dr, dashed, "\overline{\phi}"] \arrow[ddr, "\phi"'] & & \\ & W(E) \arrow[r, "p_o(E)"] \arrow[d, "W(p)"] & E \arrow[d, "p"] \\ & W(B) \arrow[r, "p_o(B)"'] & B \end{tikzcd}$$

there is a morphism $\overline{\phi} : X \longrightarrow W(E)$ with $p_o(E) \circ \overline{\phi} = f$ and $W(p) \circ \overline{\phi} = \phi$.

5.3 Adjointness and (Co)homotopy.

Let A,B be categories and $Z : A \longrightarrow B$, $W : B \longrightarrow A$ be covariant functors. We say that Z is *left adjoint* to W if there is a one-one onto morphism of sets

$$\theta(X,Y) : \mathrm{Hom}(Z(X),Y;B) \longrightarrow \mathrm{Hom}(X,W(Y);A)$$

for each $X \in |A|$, $Y \in |B|$.

θ gives two natural transformations of functors;

$$\alpha : ZW \longrightarrow 1_B$$

defined by $\alpha(Y) = \theta(W(Y),Y)^{-1}(1_{W(Y)})$, for $Y \in |B|$, and

$$\beta : 1_A \longrightarrow WZ$$

defined by $\beta(X) = \theta(X,Z(X))(1_{Z(X)})$, for $X \in |A|$.

Furthermore the functors Z and W and natural transformations α and β satisfy the following two conditions:

5.7 (i) $\alpha(Z(X)) \circ Z(\beta(X)) = 1_{Z(X)}$, $X \in |A|$

5.7 (ii) $W(\alpha(Y)) \circ \beta(W(Y)) = 1_{W(Y)}$, $Y \in |B|$.

This adjoint situation is usually summarized by writing $\theta(\alpha,\beta) : Z \dashv W\ (B,A)$.

Let A be an arbitrary category and $z = (Z,i_o,i_1,q)$ be a homotopy system in A . Assume that Z is left adjoint to some functor W . Let θ,α,β be given as above. Then following Eckmann-Hilton [4] we may construct a cohomotopy system in A as follows: Define $p_\varepsilon(X) = \alpha(X) \circ i_\varepsilon(X)$, $\varepsilon = 0,1$, and $k(X) = W(q(X)) \circ \beta(X)$, for $X \in |A|$. It

follows that $p_o, p_1 : W \longrightarrow 1_A$ and $k : 1_A \longrightarrow W$ are natural transformations of functors. The fact that $p_o \circ k = 1 = p_1 \circ k$ follows from the following lemma which we will need in the remainder of this section.

<u>Lemma 5.8</u>: For any morphisms $f : Z(X) \longrightarrow Y$ and $g : X \longrightarrow W(Y)$ in A we have

$$\text{(i)} \quad p_\varepsilon(Y) \circ W(f) \circ \beta(X) = f \circ i_\varepsilon(X) \quad , \quad \varepsilon = 0,1$$

$$\text{(ii)} \quad \alpha(Y) \circ Z(g) \circ i_\varepsilon(X) = p_\varepsilon(Y) \circ g \quad , \quad \varepsilon = 0,1 \quad .$$

<u>Proof</u> <u>(of (i))</u>: $p_\varepsilon(Y) \circ W(f) \circ \beta(X) = \alpha(Y) \circ i_\varepsilon(W(Y)) \circ w(f) \circ \beta(X) = \alpha(Y) \circ Z(W(f)) \circ i_\varepsilon(WZ(X)) \circ \beta(X)$, (i_ε is a natural transformation) $= f \circ \alpha(Z(X)) \circ Z(\beta(X)) \circ i_\varepsilon(X)$ (α, i_ε are natural transformations) $= f \circ i_\varepsilon(X)$, (5.7(i)).

<u>Corollay 5.9</u>: $w = (W, p_o, p_1, k)$ is a cohomotopy system in A .

Assume that we have a homotopy system, $z = (Z, i_o, i_1, q)$ in a category A and that Z is left adjoint to a functor W , i.e., we have an adjoint situation $\theta(\alpha,\beta) : Z \dashv W(A,A)$. This as above produces a cohomotopy system, $w = (W, p_o, p_1, k)$ in A . This then gives us the following propositions concerning the z - homotopy and w - cohomotopy relations and also about the z - fibrations and the w - fibrations.

<u>Proposition 5.10</u>: For $f,g : X \longrightarrow Y$ two morphisms in $\mathcal{A}$, $f \simeq g(z)$ if and only if $f \overline{\sim} g(w)$.

<u>Proof</u>: Assume $\phi : f \simeq g(z)$, i.e., $\phi : Z(X) \longrightarrow Y$ and $\phi \circ i_o(X) = f$, $\phi \circ i_1(X) = g$. Define $\overline{\Phi} : X \longrightarrow W(Y)$ by $\overline{\Phi} = W(\phi) \circ \beta(X)$. Then using 5.8 (i) we have $p_o(Y) \circ \overline{\Phi} = p_o(Y) \circ W(\phi) \circ \beta(X) = \phi \circ i_o(X) = f$ and $p_1(Y) \circ \overline{\Phi} = p_1(Y) \circ W(\phi) \circ \beta(X) = \phi \circ i_1(X) = g$. So $f \overline{\sim} g(w)$. The converse follows by using 5.8 (ii).

<u>Proposition 5.11</u>: A morphism $p : E \longrightarrow B$ in $\mathcal{A}$ is a z - fibration if and only if it is a w - fibration.

<u>Proof</u>: Assume $p : E \longrightarrow B$ is a z - fibration and consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[drr, "f"] \arrow[dr, dashed, "\overline{\Phi}"'] \arrow[ddr, "\phi"'] & & \\
& W(E) \arrow[r, "p_o(E)"'] \arrow[d, "W(p)"] & E \arrow[d, "p"] \\
& W(B) \arrow[r, "p_o(B)"'] & B
\end{tikzcd}$$

Now $p \circ f = p_o(B) \circ \phi = \alpha(B) \circ i_o(W(B)) \circ \phi = \alpha(B) \circ Z(\phi)$ $i_o(X)$, (i_o is a natural transformation), and this gives another commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_o(X)"'] & E \arrow[d, "p"] \\ Z(X) \arrow[r, "\alpha(B)\circ Z(\phi)"'] \arrow[ur, dashed, "\phi'"] & B \end{tikzcd}$$

Since p is a z - fibration, there exists a $\phi' : Z(X) \longrightarrow E$ with $p \circ \phi' = \alpha(B) \circ Z(\phi)$ and $\phi' \circ i_o(X) = f$. Now define $\overline{\phi} : X \longrightarrow W(E)$ by $\overline{\phi} = W(\phi') \circ \beta(X)$. Then using 5.8 (i) we have $p_o(E) \circ \overline{\phi} = p_o(E) \circ W(\phi') \circ \beta(X) = \phi' \circ i_o(X) = f$, and using 5.7 (ii) we have $W(p) \circ \overline{\phi} = W(p) \circ W(\phi') \circ \beta(X) =$ $W(p \circ \phi') \circ \beta(X) = W(\alpha(B) \circ Z(\phi)) \circ \beta(X) = W(\alpha(B)) \circ W(Z(\phi)) \circ \beta(X) =$ $W(\alpha(B)) \circ \beta(W(B)) \circ \phi = \phi$. The converse follows in a similar way using 5.7 (i) and 5.8 (ii).

<u>Example 5.12</u>: Let $A = T$, and $z = t$ as given in example 2.4. Then the Z of 2.4 is left adjoint to the W of 5.3. Using the homotopy system t and the fact that Z is left adjoint to W, we may construct a cohomotopy system w. This w is exactly the cohomotopy system described in 5.3.

<u>Example 5.13</u>: Let $C(A)$ be the category of chain complexes of an abelian category A. Then we have the homotopy system $z = (Z, i_o, i_1, q)$ in $C(A)$ defined in example 2.5. The functor Z has an adjoint W defined as follows: For $(X, \partial) \in |C(A)|$,

$\partial_n : X_n \longrightarrow X_{n-1}$, let $W(X)_n = X_n \oplus X_n \oplus X_{n+1}$ and $\delta'_n : W(X)_n \longrightarrow W(X)_{n-1}$ be given by

$$\delta'_n = \begin{pmatrix} \partial_n & 0 & 0 \\ 0 & \partial_n & 0 \\ 0 & 1_{X_n} & -\partial_{n+1} \end{pmatrix} .$$

Then $(W(X),\delta') \in |\mathcal{C}(\mathcal{A})|$. If $f : X \longrightarrow Y$ is a chain map in $\mathcal{C}(\mathcal{A})$, let $W(f) : W(X) \longrightarrow W(Y)$ be the chain map given by $W(f)_n = f_n \oplus f_n \oplus f_{n+1}$. It is easily seen that Z is left adjoint to W and thus by the above we may define a co-homotopy system, $w = (W,p_o,p_1,k)$, in $\mathcal{C}(\mathcal{A})$ such that the z - homotopy relation coincides with the w - cohomotopy relation.

We make one further application of adjointness to that of $\mathcal{M}$ - homotopy, as described in chapter IV. Assume we have an adjoint situation $\theta(\alpha,\beta) : Z \dashv W(\mathcal{B},\mathcal{A})$. Let $\mathcal{M}_\alpha$ be the class of all $\alpha(Y) : ZW(Y) \longrightarrow Y$, for $Y \in |\mathcal{B}|$, and $\mathcal{M}_\beta$ be the class of all $\beta(X) : X \longrightarrow WZ(X)$, for $X \in |\mathcal{A}|$. Let $\eta_\beta : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{M}_\beta$, $\eta_\alpha : \mathcal{B} \longrightarrow \mathcal{B}/\mathcal{M}_\alpha$, $\pi_\beta : \mathcal{A} \longrightarrow \mathcal{A}/\simeq(\mathcal{M}_\beta)$ and $\pi_\alpha : \mathcal{B} \longrightarrow \mathcal{B}/\simeq(\mathcal{M}_\alpha)$ be the natural projections.

<u>Theorem 5.14</u>: The quotient categories $\mathcal{A}/\mathcal{M}_\beta$ and $\mathcal{B}/\mathcal{M}_\alpha$ are isomorphic.

Proof: Let $\beta(X) \in M_\beta$. We claim that $(\eta_\alpha \circ Z)(\beta(X))$ is an isomorphism in $\mathcal{B}/M_\alpha$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\eta_\beta"] \arrow[dr, "\eta_\alpha \circ Z"'] & \mathcal{A}/M_\beta \arrow[d, "\Delta"] \\
 & \mathcal{B}/M_\alpha
\end{tikzcd}$$

$Z(\beta(X))$ is a morphism in $\mathcal{B}$ and by 5.7 (i) $\alpha(Z(X))$ is a left inverse for $Z(\beta(X))$ and $\alpha(Z(X))$ is in M_α . Thus by remark 4.2 we may assume that $Z(\beta(X))$ is also in M_α . Thus $\eta_\alpha(Z(\beta(X)))$ is an isomorphism in $\mathcal{B}/M_\alpha$ and by the definition of the quotient category there exists a unique $\Delta : \mathcal{A}/M_\beta \longrightarrow \mathcal{B}/M_\alpha$ with $\Delta \circ \eta_\beta = \eta_\alpha \circ Z$. Similarly there exists a unique $\Gamma : \mathcal{B}/M_\alpha \longrightarrow \mathcal{A}/M_\beta$ with $\Gamma \circ \eta_\alpha = \eta_\beta \circ Z$. Then using the uniqueness of Δ and Γ we obtain $\Delta \circ \Gamma = 1$ and $\Gamma \circ \Delta = 1$.

Corollary 5.15: The resulting homotopy categories $\mathcal{A}/\simeq(M_\beta)$ and $\mathcal{B}/\simeq(M_\alpha)$ are isomorphic.

5.4 Adjointness and Cones.

Let $z = (Z, i_o, i_1, q)$ be a homotopy system in a category $\mathcal{A}$ and assume that Z is left adjoint to a functor W , i.e., we have an adjoint situation, $\theta(\alpha,\beta) : Z \dashv W(\mathcal{A},\mathcal{A})$.

By the previous section we can construct a cohomotopy system $w = (W, p_o, p_1, k)$. The homotopy system z gives a pre cone (C, i) in A by §2.5, while the cohomotopy system gives a pre path (P, π) , (a *pre path* is a pair (P, π) , where $P : A \longrightarrow A$ is a covariant functor and $\pi : P \longrightarrow 1_A$ is a natural transformation of functors). Then the results of the previous section give:

Corollary 5.16:

(i) C is left adjoint to P ;

(ii) $\pi(X) = \alpha(X) \circ i(P(X))$, $X \in |A|$.

Corollary 5.17: A morphism $f : X \longrightarrow Y$ in $\dot{A}$ is null homotopic with respect to the pre cone (C, i) if and only if it is null homotopic with respect to the pre path (P, π) .

Remark 5.18: A cone (C, i, p) in a category A was defined in 2.20. We say that a cone (C, i, p) is a *triple* if

(i) $p \circ C(i) = 1_C$

(ii) $p \circ C(p) = p \circ p(C)$.

Dually we have the notions of *path* and *cotriple*.

If we have an adjoint situation $\theta(\alpha, \beta) : Z \dashv W(\mathcal{B}, A)$, then $(W \circ Z, \beta, W \circ \alpha \circ Z)$ is a triple in A , induced by the adjoint

situation. Also there is a cotriple $(Z \circ W, \alpha, Z \circ \beta \circ W)$ induced in $\mathcal{B}$. Furthermore, every triple and cotriple is induced in this way from some adjoint situation, as follows: Let (C,i,p) be a triple in $\mathcal{A}$ and let $C(\mathcal{A})$ be the full subcategory of $\mathcal{A}$ with objects $C(A)$ for $A \in |\mathcal{A}|$. Then $C : \mathcal{A} \longrightarrow C(\mathcal{A})$ is left adjoint to the inclusion functor $I : C(\mathcal{A}) \longrightarrow \mathcal{A}$. Then the induced triple $(I \circ C, \beta, I \circ \alpha \circ C)$ is simply (C,i,p).

For the homotopy system given in chapter II the resulting pre cones are actually not just cones but triples. The method used in constructing the Eckmann-Hilton injective homotopy system for modules was to obtain a triple and this in additive categories will be seen to be more than sufficient to determine a homotopy system.

CHAPTER VI

HOMOTOPY SYSTEMS IN ADDITIVE CATEGORIES

6.1 Introduction:

This chapter contains a detailed description of homotopy systems in additive categories. We first observe that homotopy systems are equivalent to pre cones in additive categories. This is actually a very weak condition since for any category $\mathcal{A}$, any covariant functor $C : \mathcal{A} \longrightarrow \mathcal{A}$ with $i : 1 \longrightarrow C$ given by $i(X) : X \longrightarrow C(X)$ the zero map for all $X \in |\mathcal{A}|$ determines a pre cone (C,i) . Thus we make the added requirement that the pre cone be a cone. This is not a very stringent requirement because all of the usual examples satisfy this. Also for additive categories this is equivalent to the condition that a homotopy system be a natural homotopy system. Because of this condition, it follows that the homotopy theory in additive categories is primarily concerned with the study of certain classes of objects, in particular the contractible objects, as determined by a cone. All our categories are assumed to be additive in this chapter.

6.2 Homotopy Systems.

Let $z = (Z,i_o,i_1,q)$ be a homotopy system in an additive category $\mathcal{A}$. Then since q is a retraction with coretracts

i_o, i_1, we may write Z as $1 \oplus C$, i_o as $\{1, j_o\}$, i_1 as $\{1, j_1\}$ and q as $\langle 1, r\rangle$ for a covariant functor $C : \mathcal{A} \longrightarrow \mathcal{A}$ and natural transformations of functors $j_o, j_1 : 1 \longrightarrow C$ and $r : C \longrightarrow 1$ with $r \circ j_o = 0 = r \circ j_1$. Also z may be rewritten as

$$z = (1_A \oplus C, \{1, j_o\}, \{1, j_1\}, \langle 1, r\rangle) \quad .$$

Also, given this homotopy system z, we may form a new homotopy system from it, z' by

$$z' = (1_A \oplus C, \{1, 0\}, \{1, j_1 - j_o\}, \langle 1, 0\rangle) \quad .$$

<u>Theorem 6.1</u>: For morphisms $f, g : A \longrightarrow B$ in $\mathcal{A}$, $f \simeq g(z)$ if and only if $f \simeq g(z')$.

<u>Proof</u>: Assume $\langle F, G\rangle : f \simeq g(z)$, i.e., $\langle F, G\rangle : A \oplus C(A) \longrightarrow B$ with $f = \langle F, G\rangle \circ \{1_A, j_o(A)\} = F + G \circ j_o(A)$ and $g = \langle F, G\rangle \circ \{1_A, j_1(A)\} = F + G \circ j_1(A)$. Then $g = f + G(j_1(A) - j_o(A))$. So $\langle f, G\rangle : f \simeq g(z')$.

Conversely if $\langle F, G\rangle : f \simeq g(z')$, it follows that $\langle f - G \circ j_o(A), G\rangle : f \simeq g(z)$.

Thus any homotopy system z in an additive category $\mathcal{A}$ may be written as

$$z = (1_A \oplus C, \{1, 0\}, \{1, i\}, \langle 1, 0\rangle)$$

where $C : \mathcal{A} \longrightarrow \mathcal{A}$ is a covariant functor and $i : 1_{\mathcal{A}} \longrightarrow C$ is a natural transformation of functors. (C,i) is exactly a pre cone in $\mathcal{A}$ and therefore for additive categories the study of homotopy systems is equivalent to the study of pre cones.

Theorem 6.2: Let $z = (1 \oplus C, \{1,0\}, \{1,i\}, \langle 1,0\rangle)$ be a homotopy system in an additive category $\mathcal{A}$. For morphisms $f,g : A \longrightarrow B$ in $\mathcal{A}$, $f \simeq g(z)$ if and only if $f-g$ is null homotopic with respect to the pre cone (C,i).

Proof: If $\langle F,G\rangle : f \simeq g(z)$, i.e., $\langle F,G\rangle : A \oplus C(A) \longrightarrow B$, then $f = F$ and $g = F + G \circ i(A)$. Thus $g-f = G \circ i(A)$ and so $g-f$ factors through $i(A) : A \longrightarrow C(A)$. By §2.5 $g-f$ is null homotopic with respect to the pre cone (C,i).

Conversely if $g-f = \pi \circ i(A)$, for some morphism $\pi : C(A) \longrightarrow B$, then $\langle f,\pi\rangle : f \simeq g(z)$.

Corollary 6.3: " $\simeq(z)$ " is without extension a natural equivalence relation among the morphisms in $\mathcal{A}$.

We defined in definition 4.5 a natural homotopy system and then commented on the relation between a homotopy system being natural and the resulting pre cone being a cone. For additive categories this problem is completely solved by the following proposition of Seeback [17]. The proof is computational and is omitted here.

<u>Proposition 6.4</u>: Let $z = (1_A \oplus C, \{1,0\}, \{1,i\}, \langle 1,0\rangle)$ be a homotopy system in an additive category $\mathcal{A}$. Then z is a natural homotopy system if and only if the pre cone (C,i) is actually a cone (C,i,p) .

<u>Theorem 6.5</u>: Let (C,i,p) be a cone in an additive category $\mathcal{A}$. Let $\mathcal{C} = \{C(A) : A \in |\mathcal{A}|\}$. Then the set of contractible objects in $\mathcal{A}$ is the set of all direct sums of direct factors of objects in $\mathcal{C}$.

<u>Proof</u>: If an object K in $\mathcal{A}$ is contactible, then $1_K = \pi \circ i(K)$ for some morphism $\pi : C(K) \longrightarrow K$. Thus $C(K)$ is isomorphic to $K \oplus K'$ for some object K' in $\mathcal{A}$. Thus K is a direct factor of $C(K)$, an object in $\mathcal{C}$.

The converse has two parts. First assume that an object K in $\mathcal{A}$ is a direct factor of an object in $\mathcal{C}$, i.e., there are objects A,K' in $\mathcal{A}$ with $C(A) = K \oplus K'$. Now $1_K = \langle 1_K,0\rangle \circ \{1_K,0\}$ and thus 1_K factors through $C(A)$. K is then contractible. Secondly, assume that an object in $\mathcal{A}$ is a direct sum of objects in $\mathcal{C}$, or a direct sum of two contractible ojects in $\mathcal{A}$, i.e., $K = J \oplus L$, where J and L are contactible objects in $\mathcal{A}$. Then $1_J = \pi \circ i(J)$ and $1_L = \pi' \circ i(L)$ for morphisms $\pi : C(J) \longrightarrow J$ and $\pi' : C(L) \longrightarrow L$. Now

$1_K = 1_J \oplus 1_L =$

$$(\pi \oplus \pi') \circ \langle C\{1_J,0\}, C\{0,1_L\}\rangle \circ \{C\langle 1_J,0\rangle, C\langle 0,1_L\rangle\} \circ (i(J) \oplus i(L)) .$$

$$\begin{tikzcd}[column sep=small]
& C(J \oplus L) \arrow[d, shift left, "{\langle C\{1_J,0\},C\{0,1_L\}\rangle}"] & \\
& C(J) \oplus C(L) \arrow[u, shift left, "{\{C\langle 1_J,0\rangle,C\langle 0,1_L\rangle\}}"] \arrow[dr, "\pi\oplus\pi'"] & \\
J\oplus L \arrow[ur, "i(J)\oplus i(L)"] \arrow[rr, "1_K"'] & & J\oplus L
\end{tikzcd}$$

Thus 1_K factors through $C(J\oplus L)$ and so K is contractible.

<u>Corollary 6.6</u>: If (C,i,p) is a cone in an additive category A, then a morphism is null homotopic if and only if it factors through some contractible object.

In any category with pushouts and cone (C,i), the <u>mapping</u> <u>cone</u> of a morphism $f : A \longrightarrow B$, C_f, is given by the pushout

$$\begin{tikzcd}
A \arrow[r, "i(A)"] \arrow[d, "f"'] & C(A) \arrow[d, "p_f"] \\
B \arrow[r, "f'"] & C_f
\end{tikzcd}$$

<u>Lemma 6.7</u>: Let (C,i,p) be a cone in an additive category $\mathcal{A}$. Then if a morphism $f : A \longrightarrow B$ in $\mathcal{A}$ is a homotopy equivalence, the mapping cone of f, C_f, is contractible.

<u>Proof</u>: Since $f : A \longrightarrow B$ is a homotopy equivalence, there is a morphism $g : B \longrightarrow A$ with $gf \simeq 1_A$ and $fg \simeq 1_B$. Thus by theorem 6.2 there is a morphism $\pi : C(A) \longrightarrow A$ with $g \circ f - 1_A = \pi \circ i(A)$. The mapping cone of f, C_f, may also be given by the following pushout

$$\begin{array}{ccc} A & \xrightarrow{\{f,i(A)\}} & B \oplus C(A) \\ \downarrow & & \downarrow \langle f',-p_f \rangle \\ 0 & \longrightarrow & C_f \end{array} .$$

We may form the morphism

$$F = \begin{pmatrix} 1 - f\circ g & f\circ\pi \\ -i(A)\circ g & 1 + i(A)\circ\pi \end{pmatrix} : B \oplus C(A) \longrightarrow B \oplus C(A)$$

and $F \circ \{f,i(A)\} = 0$. Thus since C_f is a pushout, there is a unique morphism $\{\delta_1,\delta_2\} : C_f \longrightarrow B \oplus C(A)$ with $\{\delta_1,\delta_2\} \circ \langle f',-P_f \rangle = F$. This means that $\delta_1 \circ f' = 1 - f\circ g$, $\delta_2 \circ f' = -i(A)\circ g$, $-\delta_1 \circ P_f = f\circ\pi$ and $-\delta_2 \circ P_f = 1 + i(A)\circ\pi$.

Also $\langle f',-P_f\rangle \circ \{\delta_1,\delta_2\} \circ \langle f',-P_f\rangle = \langle f',-P_f\rangle$ and since $\langle f',-P_f\rangle$ is an epimorphism, we have $\langle f',-P_f\rangle \circ \{\delta_1,\delta_2\} = 1_{C_f}$. Now $1_{C_f} = f'\circ\delta_1 - P_f\circ\delta_2 \simeq f'\circ\delta_1 \simeq f'\circ f\circ g\circ\delta_1 = P_f \circ i(A)\circ g\circ\delta_1 \simeq 0$. Thus C_f is contractible.

Corollary 6.8: Let (C,i,p) be a cone in an additive category $\mathcal{A}$. Then two objects A and B in $\mathcal{A}$ are homotopically equivalent if and only if there exists contractible objects K and J in $\mathcal{A}$ with $A \oplus K$ isomorphic to $B \oplus J$.

Proof: If $A \oplus K \cong B \oplus J$ for contractible objects K and J in $\mathcal{A}$, then $\{1_A,0\} : A \longrightarrow A \oplus K$ and $\langle 1_B,0\rangle : B \oplus J \longrightarrow B$ are homotopy equivalences. The composition of these three morphisms will give a homotopy equivalence from A to B .

Conversely if a morphism $f : A \longrightarrow B$ is a homotopy equivalence, then $A \oplus C_f \cong B \oplus C(A)$. This follows from the above theorem by considering the following morphisms

$$\begin{pmatrix} f & \delta_1 \\ i(A) & \delta_2 \end{pmatrix} : A \oplus C_f \rightleftarrows B \oplus C(A) : \begin{pmatrix} g & -\pi \\ f' & -P_f \end{pmatrix} .$$

6.3 Fibrations and Examples.

We assume throughout this section that $z = (1_A \oplus C,\{1,0\},\{1,i\},\langle 1,0\rangle)$ is a homotopy system, i.e., that

(C,i) is a pre cone, in an additive category $\mathcal{A}$.

<u>Lemma 6.9</u>: A morphism $p : E \longrightarrow B$ in $\mathcal{A}$ is a z - fibration if and only if for all objects X in $\mathcal{A}$, every morphism $f : C(X) \longrightarrow B$ can be lifted to E .

<u>Proof</u>: Assume $p : E \longrightarrow B$ is a z - fibration and that we have a morphism $f : C(X) \longrightarrow B$ for some object X in $\mathcal{A}$. This gives the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "0"] \arrow[d, "\{1_X{,}0\}"'] & E \arrow[d, "p"] \\
X \oplus C(X) \arrow[r, "<0{,}f>"'] & B
\end{tikzcd}$$

Since p is a z - fibration there is a homotopy $<g,g'> : X \oplus C(X) \longrightarrow E$ with $g = 0$ and $pg' = f$. So f lifts to E . The converse follows in a very similar manner.

<u>Theorem 6.10</u>: If (C,i) is a cone (C,i,p) in an additive category $\mathcal{A}$, then a morphism $p : E \longrightarrow B$ is a z - fibration if and only if it is a $\underline{\sim}(z)$ - fibration.

<u>Proof</u>: We proved in lemma 2.16 that for any homotopy system z in any category, each z - fibration is a $\underline{\sim}(z)$ - fibration. Conversely assume $p : E \longrightarrow B$ is a $\underline{\sim}$ - fibration and assume

there is a morphism $f : C(X) \longrightarrow B$ for some object X in A .

$$\begin{tikzcd} & E \arrow[d, "p"] \\ C(X) \arrow[ur, dashed, "g"] \arrow[r, "f"'] & B \end{tikzcd}$$

Since $(C,i) = (C,i,p)$, $f \simeq 0$. Then since $p \circ 0 = 0$ and p is a $\simeq$ - fibration there is a morphism $g : C(X) \longrightarrow E$ with $p \circ g = f$. By lemma 6.9 p is a z - fibration.

Thus for natural homotopy systems or cones in additive categories, two morphisms are homotopic if and only if their difference factors through some contractible object. And, a morphism is a fibration if and only if it has the homotopy lifting property if and only if it has the lifting property with respect to all contractible objects.

Example 6.11: For the trivial homotopy system of example 2.2 with cone $(0,0,1_0)$ two morphisms are homotopic if and only if they are equal. The only contractible object is the zero object and every morphism is a fibration. In the opposite trivial homotopy system of example 2.3 with cone $(1,1,1)$ all morphisms (with the same domain and range) are homotopic to each other. Every object is contractible and the fibrations are the retractions.

Example 6.12: For the Eckmann-Hilton injective homotopy system described in chapter III, the contractible objects are the injective modules. Two morphisms are homotopic if and only if their difference factors through an injective module and the fibrations are the morphisms with the lifting property with respect to any injective module.

Example 6.13: In example 2.5 and 5.13 we described a homotopy system and a cohomotopy system in the category of chain complexes of an abelian category. In these systems the contractible objects are the usual contractible chain complexes and two morphisms are homotopic iff they are chain homotopic in the usual sense.

Example 6.14: Dual to the Eckmann-Hilton injective homotopy system, there is the projective cohomotopy system or path, actually a cotriple, mentioned in example 5.4. In this system the contractible objects are the projective objects with two morphisms being homotopic if and only if their difference factors through some projective object.

6.4 Contractibility and Quotient Categories.

Let (C,i,p) be a cone in an additive category $\mathcal{A}$. Let $\mathcal{M}$ be the family of all coretractions $f : A \longrightarrow B$ in $\mathcal{A}$ such that the cokernel of f, $B/f(A)$, is contractible. This family $\mathcal{M}$ gives a quotient category, $(\mathcal{A}/\mathcal{M},\eta)$, as discussed in

chapter IV. Arising from this quotient category we have the $\mathcal{M}$ - homotopy relation where $f \simeq g(\mathcal{M})$ means $\eta(f) = \eta(g)$ for morphisms f,g in $\mathcal{A}$, and the $\mathcal{M}$ - homotopy category $\mathcal{A}/\simeq(\mathcal{M})$. For additive categories we have the following theorem.

<u>Theorem 6.15</u>: The homotopy relation as defined from the cone (C,i,p) is the same as the homotopy relation as defined from the quotient category $(\mathcal{A}/\mathcal{M},\eta)$.

<u>Proof</u>: We use theorem 4.4 to prove this. Let the cone be written as a natural homotopy system $z = (1_A \oplus C, \{1,0\}, \{1,i\}, \langle 1,0\rangle)$. Then it is necessary to prove two things: First that $\eta(\langle 1_A,0\rangle)$ is an isomorphism in $\mathcal{A}/\mathcal{M}$ for each object A in $\mathcal{A}$ and second that f is a homotopy equivalence with respect to z for all f in $\mathcal{M}$.

Now $\{1_A,0\} \in \mathcal{M}$ and so $\eta(\{1_A,0\})$ is an isomorphism with inverse say J in $\mathcal{A}/\mathcal{M}$. Now $\{1_A,0\} \circ \langle 1_A,0\rangle \circ \{1_A,0\} = \{1_A,0\}$ and so $\eta(\{1_A,0\}) \circ \eta(\langle 1_A,0\rangle) \circ \eta(\{1_A,0\}) = \eta(\{1_A,0\})$ and applying J to both sides gives $\eta(\langle 1_A,0\rangle)$ an isomorphism with inverse $\eta(\{1_A,0\})$.

Let $f : A \longrightarrow B$ be a coretraction with $B/f(A) = K$ a contractible object in $\mathcal{A}$, and with left inverse g . Then since we are in an additive category we may write $f = \{1_A,0\} : A \longrightarrow A \oplus K$ and $g = \langle 1_A,0\rangle : A \oplus K \longrightarrow A$. Now

$\langle 1_A, 0\rangle \circ \{1_A, 0\} = 1_A$, and $\{1_A, 0\} \circ \langle 1_A, 0\rangle = (1_A \oplus 0) \simeq (1_A \oplus 1_K)$, since $0 \oplus 1_K$ factors through the contractible object K . So $f = \{1_A, 0\}$ is a homotopy equivalence with respect to z .

Let N = {retractions with contractible kernel} . Then the quotient categories A/M , A/N and $A/M \cup N$ are all the same and the resulting homotopy categories are the same as that defined via the cone or the contractible objects. It also follows that the M - fibrations are the usual fibrations.

We proved in lemma 4.3 that if a family of morphisms M is a sub-family of the set of all retractions and the coretractions, then the quotient category of A by M , $(A/M, \eta)$, is the same as the resulting homotopy category determined by M , $A/\simeq(M)$. Thus the homotopy categories determined from a cone or a natural homotopy system are quotient categories or categories of fractions.

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